# Deposition of a viscous fluid on a plane surface

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Two mechanisms by which a viscous fluid can be deposited on a plane surface are described. Measurement of the thickness of the deposit are compared with calculated values. It is found that the two agree within rather wide limits of experimental error provided the effect of surface tension can be neglected, and the conditions under which this is legitimate are discussed.

## The paint brush

There are many ways in which a viscous fluid can be deposited on a plane surface. Perhaps the best known is by means of a paint brush. This consists of a number of flexible cylinders or hairs between which the fluid lies. As these hairs are dragged over the surface they lie with their axes parallel to the direction of motion and the paint is dragged through between them by tangential stress. The fluid dragged out by tangential stress acting at one side of the brush on the fluid surrounding the outermost layer of hairs must be replaced by fluid flowing transversely from the interstices between hairs further from the outer layer. This process can be understood qualitatively but to describe it mathematically is difficult. For this reason it seemed worthwhile to describe an ideal structure which would deliver fluid at a calculable rate even though it has no practical use in the art of painting.

The simplest is a plate sliding at height $h_0$ over a fixed parallel plane. If fluid is supplied at atmospheric pressure between the leading edge of the moving plate and the fixed plane, and if the length of the plane is great compared with $h_0$, the pressure in the fluid will be constant and a film of depth $\frac{1}{2}h_0$ will be left behind on the fixed plane.

If the moving plate is replaced by a portion of a cylindrical surface of any cross-section moving parallel to the generators it is possible to calculate the volume of fluid left behind. Two examples will be given.

### *Parallel plates*

An arrangement by which plates spaced at distance $2d$ apart could be made to slide on their edges over a horizontal plane is shown in figure 1. Taking an origin in the surface of the horizontal plane midway between two vertical plates, the co-ordinate $x$ is in the direction of motion, i.e. parallel to the plates and $z$ is vertical. The velocity $u$ is parallel to the direction of motion at all points and satisfies

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 0. \tag{1}$$

It is convenient to take $u$ as the velocity relative to the moving plates, so that the boundary conditions are

$$u = 0 \quad \text{at} \quad y = \pm d \quad \text{and} \quad u = U \quad \text{at} \quad z = 0.$$

The solution of (1) which satisfies these conditions is

$$u = \frac{4U}{\pi} \sum_0^\infty \frac{(-1)^n}{2n+1} \cos \frac{(2n+1)\pi y}{d} e^{-(2n+1)\pi z/2d}. \tag{2}$$

The total volume deposited on the horizontal plane per second from the fluid contained between each pair of plates is

$$Q = \int_{-d}^{+d} \int_0^\infty u\,dz\,dy = \frac{32Ud^2}{\pi^3} \sum_0^\infty \frac{1}{(2n+1)^3} = 1{\cdot}085Ud^2. \tag{3}$$

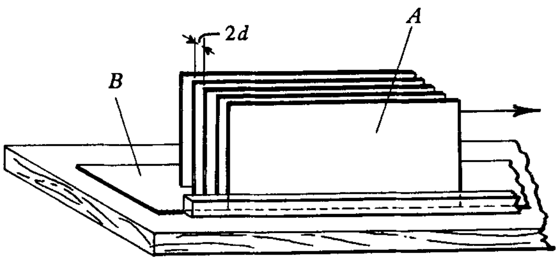


FIGURE 1. $A$, parallel plates; $B$, sheet on which deposit is collected.

*Semicircular grooves*

In the above example the way in which the fluid emerges from the spaces between the parallel plates was not considered nor was the manner in which the fluid entered them. The effect of gravity and of possible vertical motion between the plates was also assumed to be negligible. To avoid these uncertainties a 'mathematical paint brush' was designed in which the supply of fluid occurred only at the forward end of the channels which regulated the flow. A number of semicircular grooves were cut in the plane base of a Perspex block. These grooves ended at the supply end in a chamber large enough to ensure that the fluid was supplied at atmospheric pressure. There were 5 grooves each of radius 0·1 cm and they extend from the rear end of the block through a length of 6·1 cm to a cavity from which the fluid was supplied. The method of experiment is indicated in figure 2. The block $A$ was placed on a strip of very thin metal foil $B$ which had been pressed on to a steel surface plate. The block was charged with a viscous fluid, usually glycerine, through the hole $C$ which led into the supply chamber. No fluid could escape except through the grooves. The block rested against a guide so that it could move parallel to the grooves while depositing fluid on the foil. The foil was then peeled off the surface plate and a section cut from the middle using a photographic trimming knife. This section was then weighed with its deposit. After removing the deposit it was weighed again.

In a series of such experiments made with glycerine the weights deposited per cm of foil varied from 0·0422 to 0·0467 g. The mean of six runs was 0·0437 g and the corresponding volume was 0·0347 c.c. Thus the experimental value of the volume deposited per cm from each of the five grooves was 0·00694 c.c. The calculation of the volume deposited from a semicircular groove of radius $a$ was made in the same way as for the parallel plates, giving the result that the volume deposited per unit length of run was $2a^2/\pi$.* When $a = 0{\cdot}1$ cm, we have $2a^2/\pi = 0{\cdot}00638$. This is 8 % smaller than the observed value 0·00694. The discrepancy may well be due to small errors in the shaping of the grooves or to slight undetected variations in the flatness of the surface of the foil. The neglect in the calculation of the small pressure gradient along the grooves may give rise to a small error. This error, however, could be calculated and it was much less than the 8 % discrepancy, though its contribution was to increase the deposit above the calculated value $2a^2/\pi$.

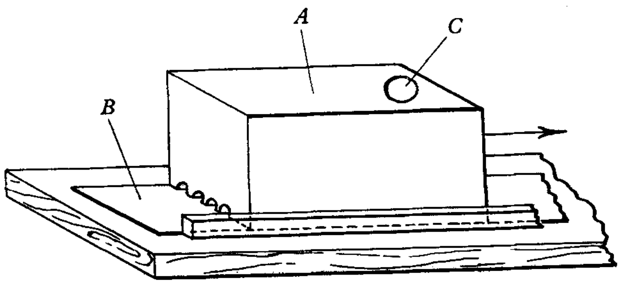


FIGURE 2. $A$, Perspex block; $B$, sheet on which fluid is deposited; $C$, filling hole.

## Porous rollers

Another way in which a viscous fluid can be deposited on a plane surface under conditions for which the flow can be described mathematically is by the use of a porous roller. This method is used by printers and sometimes by house-painters. The flow of fluid in this case has been discussed mathematically by Taylor & Miller (1956) though the connexion between the thickness of the layer deposited and the hydraulic resistance of the porous cylinder is not contained explicitly in their paper. They express their results in non-dimensional co-ordinates $x_1$ and $y$ but here $p_1$ is substituted for their $y$ as a symbol whose meaning is more obvious and

$$x_1 = x(96\kappa R^3)^{-\frac{1}{4}}, \quad p_1 = -\frac{p\kappa R}{\mu U}(96\kappa R^3)^{-\frac{1}{4}}. \tag{4}$$

Here $R$ is the radius of the cylinder, $p$ is the pressure in the fluid at distance $x$ from the point of contact, $U$ is the velocity of the roller, $\mu$ is the viscosity of the fluid and $\kappa$ is a coefficient of the dimensions of a length which occurs when Darcy's law connecting the rate of flow through a porous sheet, $W$, with $p$, is expressed in the form

$$W = \kappa p/\mu. \tag{5}$$

* Later Dr F. Ursell pointed out that the calculation could have been much simplified, because the solution of $\nabla^2 u = 0$ which satisfies $u = 0$ on the circumference and $u = U$ on the diameter is

$$u = U\left(\frac{2\theta}{\pi} - 1\right),$$

where $\theta$ is the angle subtended at the point where the velocity is $u$ by the two ends of the diameter.

Thus $\kappa$ is not the same as that which would be suitable for use with a porous medium, for the latter would be of dimensions (length)$^2$. The total rate of flow through unit length of the porous cylinder is

$$Q = \int_0^\infty W\,dx, \tag{6}$$

and on substituting from (4) and (5) we find

$$Q = (96\kappa R)^{\frac{1}{2}}\,U \int_0^\infty p_1\,dx_1. \tag{7}$$

The computed relationship between $p_1$ and $x_1$, is given in the form of a curve (figure 3, p. 134 of Taylor & Miller 1956). By graphical integration over the main part of the curve and the use of the asymptotic expression for $p_1$ at large values of $x_1$, $\int_0^\infty p_1\,dx_1$ was found to be 0·302 so that

$$Q = 2{\cdot}96U(\kappa R)^{\frac{1}{2}}. \tag{8}$$

This value for $Q$ is the value which would apply if the whole space between the porous cylinder and the plane on which it rolls were filled with fluid. In fact the fluid only passes through the porous cylindrical surface when there is suction and, at a certain point which my analysis could not determine, the flow separates at a meniscus, some fluid remaining on the flat plate and some adhering to the cylinder. Beyond this meniscus there is no further suction. In the absence of further evidence it seems that the meniscus is likely to divide the fluid equally into two streams as it would if the cylindrical surface had not been porous. Assuming this to be the case the thickness $t$ of the layer deposited would be

$$t = \frac{1}{2}\frac{Q}{U} = 1{\cdot}48(\kappa R)^{\frac{1}{2}}. \tag{9}$$

### *Measurements*

A perforated cylinder 21 cm long and 13·5 cm in diameter was made by wrapping a perforated sheet round some circular disks. These disks had central holes to permit the entry of the fluid to the inner surface of the cylinder. Two layers of flannel were wrapped round the cylinder and the seam pulled tight. The fabric was bent over the ends of the cylinder, sewn through the end perforations and sealed on the inner side to prevent fluid from escaping without passing through the flannel.

Two methods were used to measure $t$. The first was to roll the cylinder on a sheet of plate glass and then blot up the deposit with weighed sheets of blotting paper which were then rapidly enclosed in a container to prevent evaporation and weighed again with the absorbed fluid. The second, indicated in figure 3, was to use the technique employed in the experiments with the 'idealized' paint brushes described earlier, but since the thickness of the film deposited was much less than before, larger pieces of foil had to be used. Even so the blotting-paper technique proved to be more satisfactory because it was difficult to prevent the foil from being pulled off the plate on which it had been pressed as the roller passed over it. The porous roller was constructed in the manner described in order to make possible a comparison with the theoretical calculation, but even so the range of fluids for which that comparison could be made was limited.

When water was used the thickness left on the plate was sometimes very small and it was concluded that on these occasions one of the conditions assumed in the theory, that the fluid filled the space between plate and roller, was not valid and cavitation occurred. More consistent results were obtained when more viscous fluids were used, but when fluids as viscous as pure glycerine were used the suction was so great at any but very low speeds that there was danger that the flannel would leave the perforated surface on which it was stretched and thus upset the geometry of the flow.

Some of the measurements are given in table 1.

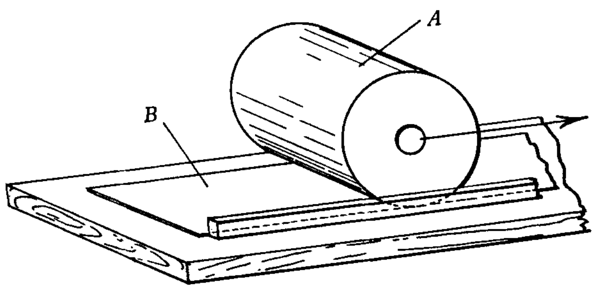


FIGURE 3. *A*, porous roller; *B*, sheet on which fluid is deposited.

| Liquid | $\mu$ (g cm$^{-1}$ sec$^{-1}$) | Film thickness ($10^{-3}$ cm) | Mean |
|---|---|---|---|
| 50 % glycerine | 0·085 | 2·39, 2·12, 1·86, 2·6, 2·44 | 2·28 |
| 30 % glycerine | 0·030 | 2·25, 1·67, 1·95, 1·77 | 1·91 |
| Water | 0·011 | 0·89, 1·28, 0·64, 1·21, 0·73 | 0·95 |

TABLE 1. Measurements of thickness of film deposited by roller.

### *Measurements of* $\kappa$

It seemed desirable to measure $\kappa$ with the flannel stretched on the perforated cylinder. The cylinder was therefore set with its axis vertical in a cylinderical vessel from which the outflow could be measured. The height $H_2$ of the fluid outside the cylinder, as well as the difference in height $H_1$ between the fluid inside and outside the porous surface, was measured. Water was used in this experiment. The volume $Q'$ flowing through the porous cylinder per second was taken as being given by

$$Q' = \frac{2\pi R\kappa}{\mu} H_1(H_2 + \tfrac{1}{2}H_1). \qquad (10)$$

It was found that when the measured values of $H_1$, $H_2$, and $Q'$ were inserted in (10) the values of $\kappa$ so found ranged from $1{\cdot}5 \times 10^{-7}$ to $3{\cdot}5 \times 10^{-7}$ cm. It seems from this large variation in $\kappa$ that flannel is not a very suitable porous material for this kind of experiment. It may be that the flow affects the geometry of the fibre structure of the flannel.

*Comparison with theory*

The theoretical values for the thickness of the deposited layer found by inserting these limiting values $1{\cdot}5 \times 10^{-7}$ and $3{\cdot}5 \times 10^{-7}$ for $\kappa$ in (9) are $1{\cdot}5 \times 10^{-3}$ and $2{\cdot}3 \times 10^{-3}$ cm. Comparison of these with the observed values given in table 1 shows that except in the case of water the agreement with the theoretical analysis is within the limit of experimental error. A possible explanation of the discrepancy in the case of water is given later.

## Effect of surface tension at the air-fluid interface

In the cases which have so far been discussed it has been tacitly assumed that the surface tension produces negligible effects. This is justifiable if the pressures due to the curvature of the free surface are small compared with those which would produce an appreciable change in the flow. In the case of the 'idealized paint brush', since the curvature of the free surface must be of the order $1/a$ (or $1/d$) for the two cases considered the pressure change on passing through the meniscus is of order $T/a$, where $T$ is the surface tension. The viscous stresses are of order $\mu U/a$. It might therefore be thought that the analysis is only realistic when $T/\mu U$ is small. This is not the case however. Taking the case illustrated in figure 2, the change in $Q$ due to change in pressure $\delta p$ between the ends of the grooves is of order $a^4 \delta p/\mu L$, where $L$ is the length of the grooves. $Q$ is of order $a^2 U$, so that the condition that a change $\delta p$ will make a negligible change in $Q$ is that $a^2 \delta p/\mu UL$ shall be small. If $\delta p$ is of order $T/a$, this condition is that $Ta/\mu UL$ shall be small.

When the measurements were made with the apparatus shown in figure 2, the influence of the surface tension was not fully appreciated so that accurate measurements of $U$ were not made. They were, however, of order $U = 4\,\text{cm/s}$. At this speed, and with glycerine, for which $\mu = 9\,\text{g}\,\text{cm}^{-1}\,\text{sec}^{-1}$ and $T = 63\,\text{g}\,\text{sec}^{-2}$, $T/\mu U = 1{\cdot}7$. This is not small, but the length of the grooves was 6 cm while the radius was 0·1 cm, so that $Ta/\mu UL = 0{\cdot}03$ and thus was sufficiently small to warrant an expectation that the calculated value of $Q$ (namely, $2a^2/\pi$) might be realized. When water was used instead of glycerine, much thinner layers were deposited.

Similar considerations apply to the porous roller. The calculations (Taylor & Miller 1956) were made assuming that the whole field of flow was flooded. In fact a meniscus or interface formed itself and divided the fluid into two streams one of which remains on the plane surface and the other is carried round on the outer surface of the roller. The meniscus is not likely to affect the distribution of suction between plane and roller unless it establishes itself within the range where the suction is appreciable. A suitable criterion for estimating whether the meniscus will have appreciable effect is to imagine that the meniscus establishes itself at the point where the calculated suction is equal to the pressure rise on passing through the meniscus due to surface tension. If this point is in the range where the suction is small compared with its maximum value, then it would not be expected to make an appreciable change in the value of $Q$.

The suction at a distance $x$ from the point of contact of roller and plane is given (Taylor & Miller 1956, p. 135) by the equation (4) and the relationship between the non-dimensional quantities $p_1$ and $x_1$ is shown in figure 3 of that paper. If the radius of curvature of the meniscus is taken as half of the distance between plane and roller at distance $x$, the meniscus will establish itself near the point where

$$\frac{4TR}{x^2} = -p. \tag{11}$$

Substitution for $x$ and $p$ from (4) and (11) gives

$$p_1 x_1^2 = \frac{T}{\mu U}(96)^{-\frac{3}{4}}\left(\frac{\kappa}{R}\right)^{\frac{1}{4}}. \tag{12}$$

When $x_1 > 1$ the asymptotic form of Taylor & Miller's expression may be used. This is

$$p_1 x_1^3 \sim \tfrac{1}{6}. \tag{13}$$

Dividing (13) by (12), an approximate value for the position of the meniscus is

$$x_1 \sim 5{\cdot}1\left(\frac{\mu U}{T}\right)\left(\frac{R}{\kappa}\right)^{\frac{1}{4}}, \tag{14}$$

where 5·1 is the approximate value of $(96)^{\frac{3}{4}}(\frac{1}{6})$.

When therefore the value of $x_1$ calculated from (14) is larger than unity, so that the corresponding value of $p_1$ is small compared with its maximum value 0·38, agreement may be expected between Taylor & Miller's calculation and the measured thickness of the deposited film of fluid. If the value calculated using (14) is less than unity the boundary condition used by Taylor & Miller in their calculation is not valid, so that agreement would not be expected. Taking values appropriate to the apparatus described, namely, $R = 6{\cdot}9$ cm and $\kappa = 2{\cdot}5 \times 10^{-7}$ cm, the value of $x_1$ at the meniscus was, according to (13),

$$x_1 \sim 3{\cdot}7 \times 10^2(\mu U/T). \tag{15}$$

In the experiments $U$ was about 3 cm/sec and $T = 62\,\mathrm{g\,sec^{-2}}$. When 50 % glycerine for which $\mu = 0{\cdot}085$ was used, $\mu U/T = 4{\cdot}1 \times 10^{-3}$ so that the value of $x_1$ at the meniscus was $(3{\cdot}7 \times 10^2)(4{\cdot}1 \times 10^{-3}) = 1{\cdot}5$. The corresponding value of $p_1$, namely, $(1{\cdot}5)^3/6$, was 0·056. This is well below the maximum value 0·378 which occurs at $x_1 = 0{\cdot}43$ (Taylor & Miller 1956) so that little difference would be expected between the amount deposited when the suction region was curtailed by a meniscus and the amount which would pass through the porous roller if the space between it and the plane were flooded.

When the working fluid is water, for which $\mu = 0{\cdot}011$ and $T = 73$ in c.g.s. units, the corresponding value of $x_1$ according to (15) would be 0·17, which is even below the value $x_1 = 0{\cdot}43$ which corresponds with the maximum value of $p_1$. Under these conditions where the suction at the meniscus is not small compared with the maximum suction, the boundary condition used by Taylor & Miller is not even approximately valid, so that the lack of agreement between the measured and calculated thickness of the deposit is understandable.

## REFERENCE

Taylor, G. I. & Miller, J. C. P. 1956 *Quart. J. Mech. Appl. Math.* **9**, 129.